Computers Math. Applic. Vol. 24, No. 1/2, pp. 119–126, 1992
0097-4943/92 $5.00 + 0.00

EXTENDED STOCHASTIC PETRI NET MODELS FOR SYSTEMS WITH PARALLEL AND COOPERATIVE MOTIONS

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Abstract—Stochastic Petri Nets have been developed to model and analyze systems involving concurrent activities with which the time associated is exponentially distributed. In this paper, we present an Extended Stochastic Petri Net that allows the firing times of its transitions to non-exponential distributions. We use it to model and analyze a multi-robot system with parallel and cooperative motions in the context of a generalized Markov Renewal Process. The modeling flexibility of Petri Net and the analyzing power of Markov Renewal Process are fully exploited.

1. INTRODUCTION

Petri Net (PN) [1] is very descriptive and flexible in modeling systems of concurrency and asynchronization. It has been developed over the last decade into a stochastic model for representing and analyzing concurrent systems, distributed systems, and reliable systems. Stochastic Petri Net (SPN) [2–5], models as well as other extensions to SPN models [6–8] have been developed to model a variety of systems in which the time associated with the occurrence of events is randomly distributed. However, SPN models can faithfully only represent systems with exponential time distributions. The usual method for analyzing an SPN model is to determine the Markov chain that is equivalent to the SPN, solve the equivalent Markov chain, and derive the desired measures for the SPN from the Markov solution. Because of this solution technique, SPN models are frequently viewed as a 'language' for specifying a Markov model of a system.

With those extended SPN models, several approaches have been suggested for incorporating non-exponential firing time distributions into SPN models. In [6], an embedded Markov chain technique was presented for the analysis of Petri Nets with both deterministic and stochastic (exponentially distributed) firing times. The use of phase-type distributions to approximate non-exponential firing times is discussed in [7]. In [8], a restricted class of SPN models with general firing time distributions, which can be solved analytically using a semi-Markov process, was defined. Their approaches are available for analytical solution with restrictive conditions or special techniques. However, they cannot been applied to analyze systems with states of both regeneration points and non-regeneration points. Unfortunately, the non-regeneration problem is very realistic in practical concurrent systems, particularly, when non-exponential time distributions are incorporated.

In this paper, we present an Extended Stochastic Petri Net (ESPN) model that allows its transitions of non-exponential firing time distributions and distinguishes the system states between regeneration points and non-regeneration points, and use it to analyze a multi-robot system with parallel and cooperative motions by a generalized Markov Renewal Process defined by Nakagawa and Osaki [9]. As is known, a Markov Renewal Process (MRP) [10,11] is a process that changes

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state according to a Markov process, but takes an arbitrarily distributed amount of time between state changes. The further development of the process depends not only on the present state, but also on the amount of time that elapsed since entry into the state. The generalized Markov Renewal Process allows that some of the MRP states may not constitute regeneration points. Nakagawa and Osaki used it in modeling and analyzing reliable systems, and showed that it is an extremely powerful analyzing tool. Markovian analysis, in principle, could be performed directly without the intervention of SPN. In practice, however, it is difficult to determine the correct Markov model for even moderately complex systems without such a descriptive 'language' like ESPN. Using ESPN, the modeling flexibility of Petri Net and the analyzing power of Markov Renewal Process can be exploited more easily. Moreover, in this paper, we introduce an Abstract Partial Reachability Graph (APRG), a sub-set of Reachability Graph, to simplify the Markov solution.

2. PETRI NET AND STOCHASTIC PETRI NET

A Petri Net is an abstract, formal graph model useful for modeling systems that exhibit concurrent, asynchronous or non-deterministic behavior. Firstly, recall the composition of a Petri Net (PN) bipartite graph [1]: a set of *places*, P (drawn as circles) representing conditions, a set of *transitions*, T (drawn as bars) representing events, and a set of *arcs*, A, which connect places to transitions or transitions to places. Places may contain *tokens* (drawn as small filled circles) denoting the conditions holding at any given time. The state of a Petri Net, called the PN *marking*, is defined by the number of tokens contained in each place.

A place is an input to a transition if an arc exists from the place to transition; a place is an output to a transition if an arc exists from a transition to the place. A transition is *enabled* when each of its input places contains at least one token. Enabled transitions can *fire*, by removing one token from each input place and depositing one token in each output place. As an event is usually enabled by a combination of conditions, a transition is enabled by a combination of tokens in places. Arcs are used to signify which combination of conditions must hold for the event to occur and which combination of conditions holds after the event occurs. Thus the firing of a transition causes a change of state (produces a different marking) for the Petri Net.

Stochastic Petri Net (SPN) [2–5] is defined by associating an exponentially distributed firing time with each transition. Once a transition is enabled, an exponentially distributed amount of time elapses. If the transition is still enabled, it will then fire. An SPN can be analyzed by considering all possible markings (enumerations of the tokens in each place) and solving the resulting reachability graph as a Markov chain. The analysis of SPN model provides information about the system it represents, provided the model is a valid representation of the system under study, and the solution of the model is correct.

3. EXTENDED STOCHASTIC PETRI NET MODEL FOR A MULTI-ROBOT SYSTEM

In this paper, as a case study, we consider a system of four robots with parallel and cooperative motions. Robots 1, 2 and 3 operate parallelly and independently. They also operate on a buffer, but at one time only one buffer operation is permitted. In addition, after Robot 3 operating on the buffer, it has to do cooperative motion with Robot 4, which is always ready (standby) to do the cooperative motion.

Figure 1 is the multi-robot system represented by Petri Net, which is referred to as an Extended Stochastic Petri Net in this paper because we allow the incorporation of arbitrary firing time distribution into it. Meanings of the places and transitions are shown as follows.

ε_1: Robot 1 is doing independent operation.
ε_2: Robot 1 is doing buffer operation.
ε_3: Robot 2 is doing independent operation.
ε_4: Robot 2 is doing buffer operation.
ε_5: Robot 3 is doing independent operation.
ε_6: Robot 3 is doing buffer operation.
ε_7: Robots 3 and 4 are doing cooperative motion.

ε_8: Robot 4 is standby.
ε_9: Buffer is idle.
ε_{10}: Buffer is busy.
τ_1: Robot 1 finishes independent operation and begins buffer operation.
τ_2: Robot 1 finishes buffer operation and begins independent operation once more.
τ_3: Robot 2 finishes independent operation and begins buffer operation.
τ_4: Robot 2 finishes buffer operation and begins independent operation once more.
τ_5: Robot 3 finishes independent operation and begins buffer operation.
τ_6: Robot 3 finishes buffer operation. Robots 3 and 4 begin cooperative motion.
τ_7: Robots 3 and 4 finish cooperative motion. Robot 3 begins independent operation once more and Robot 4 returns to standby status.

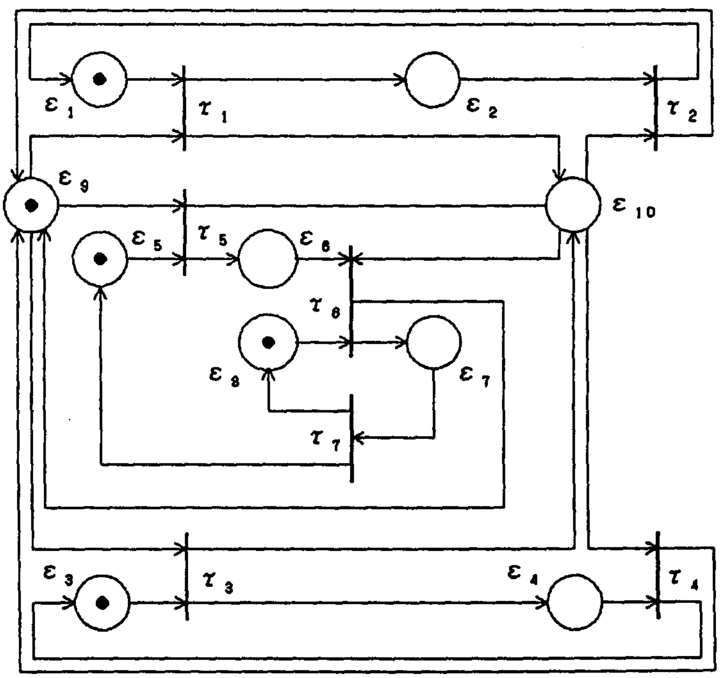


Figure 1. ESPN model of the multi-robot system.

Since disturbance may occur during Robots 3 and 4 doing their cooperative motion (i.e., the operation of transition τ_7) in this ESPN model of the multi-robot system, consider that an non-exponential distribution is associated with the firing time distribution of τ_7. For simplification of numerical solution, all the other transitions $\tau_i (i = 1, 2, \ldots, 6)$ are assumed to obey exponential firing time distribution $F_i(t) = 1 - e^{-\lambda_i t} (i = 1, 2, \ldots, 6)$.

An SPN model is generally analyzed by considering all possible marking (enumerations of the tokens in each place) and solving the resulting Reachability Graph (RG) or the marking process as a Markov chain. In this paper, for no longer all the firing times of transitions in the ESPN model obey exponential distribution, we define the marking process in terms of a renewal state space of a Markov Renewal Process, and obtain the analytic solution in the context of a generalized MRP [9]. Figure 2 is the Reachability Graph for the ESPN model in Figure 1, where the elements of the vector: 1 means a token in the corresponding place and 0 means no token. Each state $S_i (i = 0, 1, \ldots, 6)$ has the definition as follows.

S_0: Robots 1, 2 and 3 are on independent operation, Robot 4 is standby, and Buffer is idle.
S_1: Robot 1 is on buffer operation, Robots 2 and 3 are on independent operation, Robot 4 is standby, and Buffer is busy.
S_2: Robots 1 and 3 are on independent operation, Robot 2 is on buffer operation, Robot 4 is standby, and Buffer is busy.

S_3: Robots 1 and 2 are on independent operation, Robot 3 is on buffer operation, Robot 4 is standby, and Buffer is busy.

S_4: Robots 1 and 2 are on independent operation, Robots 3 and 4 are on cooperative motion, and Buffer is idle.

S_5: Robot 1 is on buffer operation, Robot 2 is on independent operation, Robots 3 and 4 are on cooperative motion, and Buffer is busy.

S_6: Robot 1 is on independent operation, Robot 2 is on buffer operation, Robots 3 and 4 are on cooperative motion, and Buffer is busy.

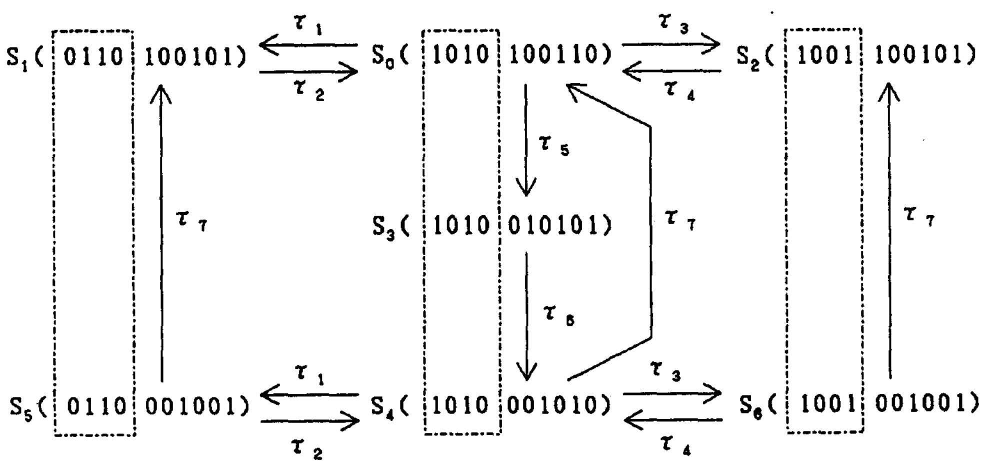


Figure 2. Reachability graph for the ESPN model in Figure 1.

4. ANALYSIS OF ESPN MODEL USING A GENERALIZED MRP

Since non-exponential distribution is incorporated to the firing time of transition τ_7 in the ESPN model, the Markovian analytic solution is different from the solution by the Markov chain and becomes more complex to solve. We introduce the Abstract Partial Reachability Graph (APRG) to simplify the Markov solution.

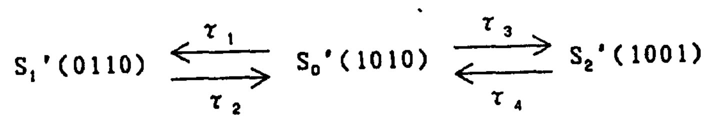


Figure 3. ARPG derived from Figure 2.

First, let us just consider the operations of Robots 1 and 2. From Figure 2, we can see S_0, S_3 and S_4 define the same state of Robots 1 and 2 on independent operations. We make them into a new state S_0'. Similarly, we find S_1 and S_5 defining the same state of Robot 1 on buffer operation and Robot 2 on independent operation, making them into a new state S_1', S_2 and S_6 defining the same state of Robot 1 on independent operation and Robot 2 on buffer operation, making them into a new state S_2'. The Reachability Graph of decreased states is re-drawn in Figure 3. It is called Abstract Partial Reachability Graph, which is equivalent to a simple Markov Renewal Process [9]. With the APRG in Figure 3, we need only to investigate the state transfers with all transitions (τ_1, τ_2, τ_3 and τ_4) of exponential firing time distributions.

For the APRG in Figure 3, transference probabilities between states S_0' and $S_j'(j = 0, 1, 2)$ are

$$p_{00}'(s) = \frac{1 - q_{01}'(s) - q_{02}'(s)}{1 - g_{00}'(s)} \tag{1}$$

$$p_{0j}'(s) = \frac{q_{0j}'(s)[1 - q_{j0}'(s)]}{1 - g_{00}'(s)}, \qquad (j = 1, 2) \tag{2}$$

where, $p_{0j}'(s)(j = 0, 1, 2)$ are the Laplace-Stieljes (LS) Transforms of $P_{0j}'(t)$, and $P_{00}'(t)$ means the probability of recurrent state transfer from S_0' to S_0';

$$g_{00}'(s) = q_{01}'(s)q_{10}'(s) + q_{02}'(s)q_{20}'(s) \tag{3}$$

is the recurrent first-passage time distribution of state S_0';

$$q_{01}'(s) = \frac{\lambda_1}{s + \lambda_1 + \lambda_3} \tag{4}$$

$$q_{02}'(s) = \frac{\lambda_3}{s + \lambda_1 + \lambda_3} \tag{5}$$

$$q_{10}'(s) = q_{20}'(s) = \frac{\lambda_b}{s + \lambda_b} \tag{6}$$

are the mass functions and here we have assumed

$$\lambda_2 = \lambda_4 = \lambda_b \tag{7}$$

Therefore, we have

$$p_{00}'(s) = \frac{s + \lambda_b}{s + \lambda_1 + \lambda_3 + \lambda_b} \tag{8}$$

$$p_{01}'(s) = \frac{\lambda_1}{s + \lambda_1 + \lambda_3 + \lambda_b} \tag{9}$$

$$p_{02}'(s) = \frac{\lambda_3}{s + \lambda_1 + \lambda_3 + \lambda_b} \tag{10}$$

Then let us go back to investigate the Reachability Graph of the ESPN model in Figure 2. From Figure 2 we see the RG forms a generalized Markov Renewal Process [9]. For we have assumed $F_i(t)(i = 1, 2, \ldots, 6)$ are exponential distributions, states S_0, S_1, S_2, S_3 and S_4 are regeneration points. On the other hand, we assume that $F_7(t)$ obeys Gamma distribution (shape parameter = 2)

$$F_7(t) = 1 - (1 + \lambda_7 t)e^{-\lambda_7 t} \tag{11}$$

Note that the firing distribution of transition τ_7, which means Robots 3 and 4 finish cooperative motion, starts from state S_4 not state S_5 or S_6. We cannot determine the start point of states S_5 and S_6. Therefore, we treat states S_5 and S_6 as non-regeneration points.

Thus, we obtain the mass functions of one-step state transference as follows

$$q_{01}(s) = \frac{\lambda_1}{s + \lambda_1 + \lambda_3 + \lambda_5} \tag{12}$$

$$q_{02}(s) = \frac{\lambda_3}{s + \lambda_1 + \lambda_3 + \lambda_5} \tag{13}$$

$$q_{03}(s) = \frac{\lambda_5}{s + \lambda_1 + \lambda_3 + \lambda_5} \tag{14}$$

$$q_{10}(s) = q_{20}(s) = q_{34}(s) = \frac{\lambda_b}{s + \lambda_b} \tag{15}$$

$$q_{40}(s) = \frac{\lambda_7^2}{\lambda_1 + \lambda_3 + \lambda_b}\left\{\frac{\lambda_b}{(s + \lambda_7)^2} + \frac{\lambda_1 + \lambda_3}{(s + \lambda_1 + \lambda_3 + \lambda_5 + \lambda_7)^2}\right\} \tag{16}$$

$$q_{41}(s) = \frac{\lambda_1\lambda_7^2}{\lambda_1 + \lambda_3 + \lambda_b}\left\{\frac{1}{(s + \lambda_7)^2} - \frac{1}{(s + \lambda_1 + \lambda_3 + \lambda_5 + \lambda_7)^2}\right\} \tag{17}$$

$$q_{42}(s) = \frac{\lambda_3\lambda_7^2}{\lambda_1 + \lambda_3 + \lambda_b}\left\{\frac{1}{(s + \lambda_7)^2} - \frac{1}{(s + \lambda_1 + \lambda_3 + \lambda_5 + \lambda_7)^2}\right\} \tag{18}$$

where

$$q_{4j}(s) = \int_0^\infty e^{-st} P_{0j}' dF_7(t), \qquad (j = 0, 1, 2) \tag{19}$$

Next, we define one state $S_i(i = 0, 1, 2, 3, 4)$ of the system at time t as that when, after the system enters this state at a time point, it will not transfer to a new state until time t. Define that the probabilities of system state transference $P_{0i}(t)(i = 0,1,2,3,4)$ are the probabilities that the system starts from state S_0 at time 0, enters and stays at state S_i. Their LS transforms are

$$p_{00}(s) = \frac{1 - q_{01}(s) - q_{02}(s) - q_{03}(s)}{1 - g_{00}(s)} \tag{20}$$

$$p_{01}(s) = \frac{[q_{01}(s) + q_{03}(s)q_{34}(s)q_{41}(s)][1 - q_{01}(s)]}{1 - g_{00}(s)} \tag{21}$$

$$p_{02}(s) = \frac{[q_{02}(s) + q_{03}(s)q_{34}(s)q_{42}(s)][1 - q_{02}(s)]}{1 - g_{00}(s)} \tag{22}$$

$$p_{03}(s) = \frac{q_{03}(s) - q_{03}(s)q_{34}(s)}{1 - g_{00}(s)} \tag{23}$$

$$p_{04}(s) = \frac{[q_{03}(s)q_{34}(s)][1 - q_{40}(s) - q_{41}(s) - q_{42}(s)]}{1 - g_{00}(s)} \tag{24}$$

where

$$g_{00}(s) = q_{01}(s)q_{10}(s) + q_{02}(s)q_{20}(s) + q_{03}(s)q_{34}(s)[q_{40}(s) + q_{41}(s)q_{10}(s) + q_{42}(s)q_{20}(s)] \tag{25}$$

is the recurrent first-passage time distribution of state S_0.

Recalling the steady-state probabilities of state $S_i(i = 0,1,2,3,4)$

$$P_i = \lim_{t\to\infty} P_{0i}(t) = \lim_{s\to 0} p_{0i}(s) \tag{26}$$

we obtain

$$P_0 = \lambda_1 + \lambda_3 + \lambda_5 \frac{1}{l_{00}} \tag{27}$$

$$P_1 = \frac{1}{\lambda_6}\left\{\frac{\lambda_1}{\lambda_1+\lambda_3+\lambda_5} + \frac{\lambda_5}{\lambda_1+\lambda_3+\lambda_5}\,\frac{\lambda_1}{\lambda_1+\lambda_3+\lambda_6}\left[1 - \frac{\lambda_7^2}{(\lambda_1+\lambda_3+\lambda_6+\lambda_7)^2}\right]\right\}\frac{1}{l_{00}} \tag{28}$$

$$P_2 = \frac{1}{\lambda_6}\left\{\frac{\lambda_3}{\lambda_1+\lambda_3+\lambda_5} + \frac{\lambda_5}{\lambda_1+\lambda_3+\lambda_5}\,\frac{\lambda_3}{\lambda_1+\lambda_3+\lambda_6}\left[1 - \frac{\lambda_7^2}{(\lambda_1+\lambda_3+\lambda_6+\lambda_7)^2}\right]\right\}\frac{1}{l_{00}} \tag{29}$$

$$P_3 = \left\{\frac{1}{\lambda_6}\,\frac{\lambda_5}{\lambda_1+\lambda_3+\lambda_5}\right\}\frac{1}{l_{00}} \tag{30}$$

$$P_4 = \left\{\frac{2}{\lambda_7}\,\frac{\lambda_5}{\lambda_1+\lambda_3+\lambda_5}\right\}\frac{1}{l_{00}} \tag{31}$$

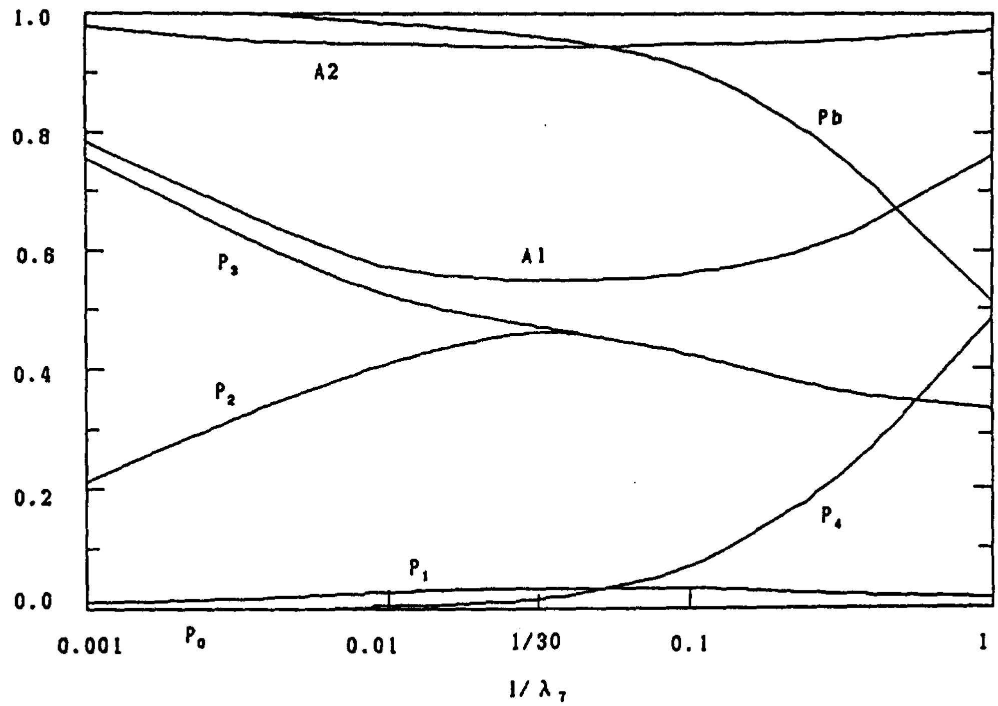
where

$$l_{00} = \frac{1}{\lambda_6} + \frac{1}{\lambda_1+\lambda_3+\lambda_5} + \frac{\lambda_5}{\lambda_1+\lambda_3+\lambda_5}\left\{\frac{2}{\lambda_7} + \frac{\lambda_1+\lambda_3}{\lambda_6(\lambda_1+\lambda_3+\lambda_6)}\left[1 - \frac{\lambda_7{}^2}{(\lambda_1+\lambda_3+\lambda_6+\lambda_7)^2}\right]\right\} \tag{32}$$

is the average recurrent time at state S_0. It is evident that

$$\sum_{i=0}^{4} P_i = 1. \tag{33}$$

From the Reachability Graph in Figure 2 and the state definitions, we can see that $A1 = P_0 + P_2 + P_3 + P_4$ depicts the Availability of Robot 1 on independent operation. Similarly, $A2 = P_0 + P_1 + P_3 + P_4$ is the Availability of Robot 2 on independent operation. The Availability of Robots 3 and 4 on cooperative motion is represented by P_4. P_4 also depicts the ratio of competition between Robots 1 and 2, while the ratio of competition among Robots 1, 2 and 3 is given by P_0. Moreover, the Performability of the buffer is described by $Pb = P_1 + P_2 + P_3$.



$1/\lambda_1 = 0.1 \quad 1/\lambda_2 = 1/\lambda_4 = 1/\lambda_6 = 1 \quad 1/\lambda_3 = 0.01 \quad 1/\lambda_5 = 0.001$

Figure 4. Numerical results of the steady-state probabilities.

5. NUMERICAL RESULTS

We show a numerical example when real time distributions for the firing of transitions in the ESPN model are given. Assume that the mean independent operation times of Robots 1, 2 and 3 are $1/\lambda_1 = 0.1$ second, $1/\lambda_3 = 0.01$ second and $1/\lambda_5 = 0.001$ second, respectively. All the mean times of buffer operation are the same, i.e., $1/\lambda_2 = 1/\lambda_4 = 1/\lambda_6 = 1/\lambda_b = 1$ second. Moreover, $1/\lambda_7$, the mean cooperative motion time of Robots 3 and 4, is variable from 0.001 to 1 second.

Figure 4 gives the steady-state probabilities P_i $(i = 0,1,2,3,4)$ and A1, A2 and Pb. It is shown from Figure 4 that the Performability of the buffer Pb decreases with $1/\lambda_7$. When $1/\lambda_7$ is smaller than 1/30 second, Pb is very close to 1, which means that the buffer in this multi-robot system has a good performability. However, if it is more than 1/30 second, the Performability of the buffer Pb is worsened sharply. P_4, the Availability of Robots 3 and 4 on cooperative motion, increases with $1/\lambda_7$, which is consistent with the fact that the number of times to do cooperative motion is varied with the speed-up of cooperative motion. Both P_1 and P_2 have their maximum values at $1/\lambda_7 = 1/30$ second, while the Availability of Robots 1 and 2 on independent operations A1 and A2 are shaped like a valley. This shows that a balance of independent operation rate has a great effect to the efficiency of the multi-robot system. It is interesting that P_2 and P_3 are almost the same when $1/\lambda_7$ varies from 1/30 to 1 second. One of the most important results is that P_0, the ratio of competition among Robots 1, 2 and 3 to do buffer operation, is almost zero (at the level of 10^{-4}), which means that the competition of buffer operation does not become a problem for the system.

6. CONCLUSIONS

In this paper, we have developed an Extended Stochastic Petri Net model that allows firing times to non-exponential distributions, and presented an analysis method using a generalized Markov Renewal Process. We applied it to evaluate performance of a multi-robot system with parallel and cooperative motions. Our main contribution is to provide an available method to numerically solve ESPN model of concurrent systems incorporated with non-exponential distributions using a generalized MRP in which non-regeneration points may be included. The suitability, availability and flexibility of the method for concurrent systems have been proven by the case study. The numerical results presented in this paper display the superiority over the works in [6–8]. We have also shown it is possible that, by cleverly exploiting Abstracted Partial Reachability Graph, the solution of larger model can be obtained with acceptable complexity.

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